Malaysian Journal of Mathematical Sciences 20(3): 1097–1119 (2026)
https://doi.org/10.47836/mjms.20.3.11

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Approximating Fixed Points of Multi-Valued Generalized Nonexpansive Mappings with Applications

Yadav, S.*,[1] and Chamola, B. P.[1]

[1]*Jaypee Institute of Information Technology, Noida, 201309, Uttar Pradesh, India*

E-mail: suchetaydv@gmail.com
**Corresponding author*

Received: 9 September 2025
Accepted: 22 January 2026

Abstract

In this paper, we extend the concept of generalized $(\alpha, \beta)-$ nonexpansive type 1 mappings to the multivalued setting. We establish several fundamental properties of mappings of this class and derive fixed point existence results under suitable assumptions in Banach spaces. To approximate their common fixed points in the context of Banach spaces, we give a multivalued variant of the M-iterative process. The convergence and stability results are establish, supported by illustrative numerical examples. These results broaden the scope of iterative approximation techniques and contribute to the growing theory of multi-valued nonexpansive-type mappings. The study further includes an application to Volterra-type integral inclusions.

Keywords: common fixed point, uniformly convex Banach space, strong convergence, stability.

1 Introduction

Fixed point theory for multivalued mappings is a foundational topic in mathematics, offering powerful tools for solving problems in diverse scientific and engineering domains. Górniewicz [10] developed the topological framework of fixed point theory for multivalued mappings. Multivalued maps as effective tools in mathematical modeling and rigorous numerics were highlighted by Kaczyński [16]. Strong convergence results for fixed points of multivalued mappings in Hadamard spaces were established by Salisu et al. [30]. More recently, Thangaraj et al. [40] demonstrated applications of multivalued fixed point theory in the generation of fractals via iterated function systems. The foundation of this theory lies in classical fixed point theorems, such as those of Brouwer [8] and Banach [6], which have been extended for multivalued mappings. Markin [18] in the Banach spaces and Nadler [20] in complete metric spaces extended the Banach fixed-point theorem to multivalued mappings. Kakutani's [17] theorem extends Brouwer's result by considering upper semicontinuous multivalued maps with compact convex values. Multivalued mappings connect each point in the domain to a set of points in the codomain, introducing a higher level of complexity and depth to the analysis.

In recent years, significant progress has been made in developing efficient iterative schemes for single-valued mappings that complement and inspire similar investigations in the multivalued setting. Iterative processes such as the D* [13], AG [15], F* [14], D-plus [4] and Picard-like schemes [21] have demonstrated improved convergence and stability characteristics, while others have addressed split fixed point problem with multiple output sets in CAT(0) spaces [28] and iterative fixed-point schemes in digital metric spaces [34]. These advancements in constructing faster and more stable iterative methods provide valuable insights that motivate the extension of such ideas to multivalued mappings, where the analysis becomes more intricate due to the set-valued nature of the operators.

Multivalued analogues of generalized single-valued mappings have been studied extensively: Suzuki [39] established fixed point and convergence theorems for generalized nonexpansive mappings, Aoyama and Kohsaka [5] studied $\alpha-$nonexpansive mappings, Pant and Shukla [24] proposed iterative schemes for approximating fixed points of generalized $\alpha-$nonexpansive mappings, Abkar and Eslamian [2] extended these results to multivalued mappings, and Sadhu and Nahak [29] further investigated common fixed points of generalized $\alpha-$nonexpansive multivalued mappings using modified S-type iteration. Numerous authors have explored iterative methods for approximating fixed points of multivalued mappings, employing a range of iterative schemes. Sastry and Babu [31] established the convergence of multivalued nonexpansive mappings to fixed points in Hilbert spaces using Mann and Ishikawa iteration methods. Building on their work, Panyanak [25] extended these results to the more general setting of uniformly convex Banach spaces. Song [38] strengthened the study of iterative processes for multivalued mappings by introducing a stronger requirement, known as the endpoint condition, to ensure convergence in Banach spaces. Shahzad and Zegeye [35] then extended this framework to multivalued quasi-nonexpansive mappings using the projection operator $\mathbb{P}_\Psi u = \{v \in \Psi u : ||u - v|| = d(u, \Psi u)\}$. Shaddad et al. [33] derived fixed point results for multivalued mappings in cone metric spaces, generalizing classical metric space results. Patel et al.[26] derived fixed point theorems for distinct mappings under novel rational contraction frameworks that employ altering distance functions within Hilbert spaces. For the study of multivalued versions of well-known iterative schemes, Abdeljawad et al. [1] investigated iterative approximation of endpoints for multivalued mappings in Banach spaces. Noor-type iteration schemes for multivalued mappings in CAT(0) spaces were analyzed by Pathak et al. [27].

Akutsah and Narain [3] introduced a broad class of single-valued nonexpansive mappings, referred to as generalized (α, β) nonexpansive type-1 mappings, motivated by which, in this paper, we investigate the class of multivalued generalized (α, β) nonexpansive type-1 mappings. The paper is structured as follows: In Section 2, we go over the preliminaries and necessary terminology. Multivalued generalized $(\alpha, \beta)-$nonexpansive type-1 mappings are defined and fixed point results are investigated within the settings of Banach spaces in Section 3. Furthermore, a multivalued extension of the M-iterative procedure is presented for convergence and approximation of common fixed point of three such mappings. Additionally, stability results are obtained for these mappings and supporting numerical examples are provided to illustrate the suggested approach's convergence behaviour. Finally, the existence of a solution to the system of Volterra-type integral inclusion equations is shown in Section 4 using our proposed iterative process.

2 Preliminaries

For clarity and ease of reference, we first outline the notations used throughout the manuscript.

Notation	**Meaning**
$(M, \|.\|)$	Banach space
T	nonempty subset of M
CB(T)	set of all nonempty bounded closed subsets of M
CP(T)	set of all nonempty compact subsets of M
KC(T)	set of all nonempty compact convex subsets of M
P(T)	set of all nonempty proximinal bounded subsets of M
Ψ	multivalued mapping
u, v, w	arbitrary points of the set T
$\mathbb{P}_\Psi$	projection operator

A set $T \subset M$ is proximinal in a normed linear space M if for every $u \in M$, there exists $v \in T$ satisfying $\|u - v\| = d(u, T)$, $d(u, T) = inf\ \|u - v\| : v \in T$.
Let $P, Q \in CB(T)$, then

$$H(P, Q) = max \left\{ \sup_{u \in P} d(u, Q), \sup_{u \in Q} d(u, P) \right\}$$

defines the Pompeiu-Hausdorff distance between the sets P and Q. A point $\mathrm{g} \in T$ is called a fixed point of a multivalued mapping $\Psi : T \to CB(T)$ if $\mathrm{g} \in \Psi(\mathrm{g})$. Let $\Psi : M \to 2^M$ be a multivalued map on a complete metric space (M, d) such that for every $u, v \in M$, $H(\Psi(u), \Psi(v)) \leq \theta d(u, v)$ for $\theta \in [0, 1)$, where H is the Hausdroff distance. Then, $\exists g \in M$ such that $g \in \Psi(u)$. The multivalued versions of several of the generalized single-valued mappings given in the literature are given below.

Definition 2.1. *A multivalued mapping $\Psi : T \to CB(T)$ is said to be*

(i) *contraction* [20] *if* $H(\Psi u, \Psi v) \leq \theta \|u - v\|$, *for all* $u, v \in T, \theta \in [0, 1)$,

(ii) *nonexpansive* [9] *if* $H(\Psi u, \Psi v) \leq \|u - v\|$, *for all* $u, v \in T$,

(iii) *quasi-nonexpansive* [36], *if* $F(\Psi) \neq \emptyset$ *and* $H(\Psi u, \Psi \mathrm{g}) \leq \|u - \mathrm{g}\|$, *for all* $u \in T, \mathrm{g} \in F(\Psi)$, *where* $F(\Psi)$ *is the set of all fixed points of the map* Ψ, (*iv*) *Suzuki generalized nonexpansive* [2] *if for* $u, v \in T$,

$$\frac{1}{2}d(u, \Psi u) \leq \|u - v\| \Rightarrow H(\Psi u, \Psi v) \leq \|u - v\|,$$

(*v*) $\alpha-$*nonexpansive if there exists* $\alpha < 1$, *such that for all* $u, v \in T$,

$$H(\Psi u, \Psi v)^2 \leq \alpha d(u, \Psi v)^2 + \alpha d(v, \Psi u)^2 + (1 - 2\alpha)\|u - v\|^2,$$

(*vi*) *generalized* $\alpha-$*nonexpansive mapping* [29] *if there exists* $\alpha < 1$, *such that for all* $u, v \in T$,

$$\frac{1}{2}d(u, \Psi u) \leq \|u - v\|$$

$$\Rightarrow H(\Psi u, \Psi v) \leq \alpha d(u, \Psi v) + \alpha d(v, \Psi u) + (1 - 2\alpha)\|u - v\|.$$

Definition 2.2. [3] *A mapping* $\Psi : T \to T$ *defined on a nonempty closed convex subset* T *of a real B.S.* M *is said to be generalized* $(\alpha, \beta)-$*nonexpansive type-1 mapping , if there exists* $\alpha, \beta, \eta \in [0, 1)$ *with* $\alpha \leq \beta$ *and* $\alpha + \beta < 1$ *such that for all* $u, v \in T$,

$$\eta \|\Psi u - \Psi v\| \leq \|u - v\|$$

$$\Rightarrow \|\Psi u - \Psi v\| \leq \alpha \|v - \Psi u\| + \beta \|u - \Psi v\| + (1 - (\alpha + \beta)) \|u - v\|.$$

Definition 2.3. [22] *"A B.S.* M *is said to have Opial property if for every weakly convergent sequence* $\{u_n\}$ *in* M *with weak limit* v, *we have*

$$\liminf_{n\to\infty} \|u_n - v\| < \liminf_{n\to\infty} \|u_n - w\|, \forall w \in M." \tag{1}$$

3 Main Results

3.1 Multivalued generalized α, β nonexpansive type-1 mapping

In this section, we extend Definition 2.2 to the multivalued mappings as follows.

Definition 3.1. *Let* $(M, \|.\|)$ *be a B.S. and* T *be a subset (nonempty) of* M. *A mapping* $\Psi : T \to CB(T)$ *is said to be multivalued generalized* $(\alpha, \beta)-$*nonexpansive type-1 (MGN* (α, β)*) mapping, if* $\exists\ \alpha, \beta \in [0, 1), \eta \in [0, 1]$ *with* $\beta \leq \alpha$ *and* $\alpha + \beta < 1$, *such that* $\forall u, v \in T$,

$$\frac{\eta}{2}d(\Psi u, u) \leq \|u - v\|$$

$$\Rightarrow H(\Psi u, \Psi v) \leq \alpha d(v, \Psi u) + \beta d(u, \Psi v) + (1 - (\alpha + \beta)) \|u - v\|. \tag{2}$$

Proposition 3.1. *The following assertions follow from the above definition:*

(*i*) *Every mapping satisfying condition* (*iv*) *of Definition* 2.1 *is an MGN* (α, β) *mapping.*

(*ii*) *Every mapping satisfying condition* (*vi*) *of Definition* 2.1 *is an MGN* (α, β) *mapping.*

(*iii*) *Every MGN* (α, β) *mapping with* $F(\Psi) \neq \emptyset$ *is a multivalued quasi nonexpansive map.*

Proof.

(i) If $\alpha = \beta = 0$ and $\eta = 1$ in (2), then the MGN (α, β) mapping satisfies condition 2.1(iv).

(ii) If $\alpha = \beta$ and $\eta = 1$ in (2), then the MGN (α, β) mapping satisfies the condition 2.1(vi).

(iii) Since $F(\Psi) \neq \emptyset$, let $\mathrm{g} \in F(\Psi)$. Therefore,

$$\frac{\eta}{2} d(\Psi \mathrm{g}, \mathrm{g}) < d(\Psi \mathrm{g}, \mathrm{g}) = 0 \leq \|u - \mathrm{g}\|, \text{for all } u \in T.$$

Since Ψ is an MGN (α, β) mapping, $\exists\, \alpha, \beta \in [0,1)$ such that

$$\begin{aligned} H(\Psi u, \Psi \mathrm{g}) &\leq \alpha d(\mathrm{g}, \Psi u) + \beta d(u, \Psi \mathrm{g}) + (1 - (\alpha + \beta)) \|u - \mathrm{g}\| \\ &\leq \alpha H(\Psi \mathrm{g}, \Psi u) + \beta \|u - \mathrm{g}\| + \beta d(\mathrm{g}, \Psi \mathrm{g}) + (1 - (\alpha + \beta)) \|u - \mathrm{g}\| \end{aligned}$$

$$\Rightarrow (1 - \alpha) H(\Psi u, \Psi \mathrm{g}) \leq (1 - \alpha) \|u - \mathrm{g}\|$$

Since, $\alpha \in [0, 1)$, it follows $H(\Psi u, \Psi \mathrm{g}) \leq \|u - \mathrm{g}\| \; \forall\, u \in T$ and $\mathrm{g} \in F(\Psi)$ i.e., Ψ is a multivalued quasi nonexpansive mapping.

□

The following example shows that the converse of above statements are not always true.

Example 3.1. *Let $T = [0, 6]$ be a subset of $\mathbb{R}$ with usual norm. Define a multivalued mapping $\Psi : T \to CB(T)$ as*

$$\Psi(u) = \begin{cases} [0, \frac{u}{3}] & \text{if } u \neq 6, \\ [0, 3] & \text{if } u = 6. \end{cases}$$

Then, for the pairs (u, v) such that $u = 6, v \in \left(4, \frac{24}{5}\right]$ and $u \in \left(4, \frac{36}{7}\right]$ and $v = 6$, $\frac{1}{2} d(u, \Psi u) \leq \|u - v\|$, but $H(\Psi u, \Psi v) > \|u - v\|$, therefore the mapping Ψ does not satisfy Suzuki generalized nonexpansive condition. But Ψ satisfies MGN (α, β) mapping condition for $\frac{1}{19} \leq \beta \leq \alpha \leq \frac{1}{3}$ and $\alpha + \beta \leq \frac{2}{3}$.

Example 3.2. *Let $T = [0, \infty)$ be a subset of $\mathbb{R}$ with usual norm. Define a multivalued mapping $\Psi : T \to CB(T)$ as*

$$\Psi(u) = \begin{cases} [0, \frac{u}{2}] & \text{if } u \in [0, \infty) \setminus \{1\} = A, \\ [0, \frac{7}{10}] & \text{if } u = 1. \end{cases}$$

Then, we verify that $\mathbb{P}_{\Psi}$ is an MGN (α, β) map but is not a generalized $\alpha-$nonexpansive mapping.

If $u \in A$, then $\mathbb{P}_{\Psi} u = \{\frac{u}{2}\}$ and if $u = 1$, then $\mathbb{P}_{\Psi} u = \{\frac{7}{10}\}$. We will prove that $\mathbb{P}_{\Psi}$ is an MGN (α, β) map for $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, \eta = 1$.

Case 1: When $u = v = 1$,

$$\frac{\eta}{2} d(u, \mathbb{P}_{\Psi} u) = \frac{3}{20} \nleq 0 = |u - v|,$$

$\therefore$ MGN (α, β) condition need not be checked for this case.

Case 2: When $u, v \in A$, then, for all u, v such that

$$\frac{\eta}{2} d(u, \mathbb{P}_\Psi u) = \frac{1}{4} u \leq |u - v|,$$

$$\begin{aligned}
\alpha d(v, \mathbb{P}_\Psi u) + \beta d(u, \mathbb{P}_\Psi v) + (1 - (\alpha + \beta)||u - v|| &= \frac{1}{3} d(v, \mathbb{P}_\Psi u) + \frac{1}{4} d(u, \mathbb{P}_\Psi v) + \frac{5}{12}|u - v| \\
&= \frac{1}{3}|v - \frac{u}{2}| + \frac{1}{4}|u - \frac{v}{2}| + \frac{5}{12}|u - v| \\
&= \frac{1}{6}|u - 2v| + \frac{1}{8}|2u - v| + \frac{5}{12}|u - v| \\
&\geq \frac{1}{8}|u - 2v| + \frac{1}{8}|2u - v| + \frac{5}{12}|u - v| \\
&\geq \frac{3}{8}|u - v| + \frac{5}{12}|u - v| = \frac{19}{24}|u - v| \\
&\geq \frac{1}{2}|u - v| = H(\mathbb{P}_\Psi u, \mathbb{P}_\Psi v).
\end{aligned}$$

Case 3: When $u = 1, v \in A$, then, for all u, v such that

$$\frac{\eta}{2} d(u, \mathbb{P}_\Psi u) = \frac{1}{4} u \leq |1 - v|,$$

$$\begin{aligned}
\frac{1}{3} d(v, \mathbb{P}_\Psi u) + \frac{1}{4} d(u, \mathbb{P}_\Psi v) + \frac{5}{12}|u - v| &= \frac{1}{3}|v - \frac{7}{10}| + \frac{1}{4}|1 - \frac{v}{2}| + \frac{5}{12}|1 - v| \\
&\geq \left|\frac{9}{10} - \frac{7}{8} v\right| \geq \left|\frac{7}{10} - \frac{v}{2}\right| = H(\mathbb{P}_\Psi u, \mathbb{P}_\Psi v).
\end{aligned}$$

Case 4: When $u \in A, v = 1$, then, for all u, v such that

$$\frac{\eta}{2} d(u, \mathbb{P}_\Psi u) = \frac{1}{4} u \leq |u - 1|,$$

$$\begin{aligned}
\frac{1}{3} d(v, \mathbb{P}_\Psi u) + \frac{1}{4} d(u, \mathbb{P}_\Psi v) + \frac{5}{12}|u - v| &= \frac{1}{3}|1 - \frac{x}{2}| + \frac{1}{4}|x - \frac{7}{10}| + \frac{5}{12}|u - 1| \\
&\geq \left|\frac{5}{6} u - \frac{11}{120}\right| \geq \left|\frac{u}{2} - \frac{7}{10}\right| = H(\mathbb{P}_\Psi u, \mathbb{P}_\Psi v).
\end{aligned}$$

$\therefore$ $\mathbb{P}_\Psi$ is an MGN (α, β) map for $\alpha = \frac{1}{3}, \beta = \frac{1}{4}$.

But, For $u = 1, v = 0.1$, we have

$$\frac{1}{2} d(u, \mathbb{P}_\Psi u) = \frac{3}{20} \leq \frac{9}{10} = |u - v|.$$

However,

$$\begin{aligned}
\Psi(\mathbb{P}_\Psi u, \mathbb{P}_\Psi v) = \frac{13}{20} > \frac{37}{60} &= \alpha d(v, \mathbb{P}_\Psi u) + \alpha d(u, \mathbb{P}_\Psi v) + (1 - (2\alpha)||u - v|| \\
&= \frac{1}{3}.0 + \frac{1}{3}|1 - \frac{1}{20}| + \frac{1}{3}.|1 - \frac{1}{10}|.
\end{aligned}$$

$\therefore$ $\mathbb{P}_\Psi$ is not a generalized alpha nonexpansive mapping.
The next result describes the behavior of MGN (α, β) mapping through a set of fundamental inequalities, forming the basis for subsequent convergence and fixed point results.

Lemma 3.1. *Let $(M, \|.\|)$ be a B.S., T be a subset(nonempty) of M and $\Psi : T \to CB(T)$ be an MGN (α, β) map. Then, for each $u, v \in T$, $\exists\, \alpha, \beta, \in [0, 1), \eta \in [0, 1]$ with $\beta \leq \alpha$ and $\alpha + \beta < 1$, the following are true:*

(*i*) $H(\Psi u, \Psi w) \leq \|u - w\|, \forall w \in \Psi u$;

(*ii*) *either* $\frac{\eta}{2} d(u, \Psi u) \leq \|u - v\|$ *holds true or* $\frac{\eta}{2} d(w, \Psi w) \leq \|w - v\|, \forall w \in \Psi u$;

(*iii*) *either* $H(\Psi u, \Psi v) \leq \alpha d(v, \Psi u) + \beta d(u, \Psi v) + (1 - (\alpha + \beta)) \|u - v\|$ *or* $H(\Psi w, \Psi v) \leq \alpha d(v, \Psi w) + \beta d(w, \Psi v) + (1 - (\alpha + \beta)) \|w - v\|, \forall w \in \Psi u$.

Proof.

(i) Given $\eta \in [0, 1]$, we have

$$\frac{\eta}{2} d(u, \Psi u) < d(u, \Psi u) \leq \|u - w\|, \text{for all } w \in \Psi u$$

Then from definition (3.1), we get

$$\begin{aligned} H(\Psi u, \Psi w) &\leq \alpha d(w, \Psi u) + \beta d(u, \Psi w) + (1 - (\alpha + \beta)) \|u - w\| \\ &= \beta d(u, \Psi w) + (1 - (\alpha + \beta)) \|u - w\| \\ &\leq \beta \|u - w\| + \beta d(w, \Psi w) + (1 - (\alpha + \beta)) \|u - w\| \\ &= \beta d(w, \Psi w) + (1 - \alpha) \|u - w\| \\ &\leq \beta d(w, \Psi u) + \beta H(\Psi u, \Psi w) + (1 - \alpha) \|u - w\| \\ &= \beta H(\Psi u, \Psi w) + (1 - \alpha) \|u - w\| \end{aligned}$$

$$\Rightarrow H(\Psi u, \Psi w) \leq \beta H(\Psi u, \Psi w) + (1 - \alpha) \|u - w\|.$$

which gives,
$$H(\Psi u, \Psi w) \leq \frac{(1 - \alpha)}{(1 - \beta)} \|u - w\| \leq \|u - w\|.$$

(ii) We establish the result using a contradiction approach. Let

$$\frac{\eta}{2} d(u, \Psi u) > \|u - v\| \text{ and } \frac{\eta}{2} d(w, \Psi w) > \|w - v\|$$

from (i), we obtain

$$\begin{aligned} d(w, \Psi w) \leq H(\Psi u, \Psi w) &\leq \|u - w\| \\ &\leq \|u - v\| + \|v - w\| < \frac{\eta}{2} d(u, \Psi u) + \frac{\eta}{2} d(w, \Psi w) \\ \Rightarrow d(w, \Psi w) &< \frac{\eta}{(2 - \eta)} d(u, \Psi u) < d(u, \Psi u). \end{aligned}$$

Since,

$$\begin{aligned} d(u, \Psi u) &\leq \|u - w\| \leq \|u - v\| + \|v - w\| \\ &< \frac{\eta}{2} d(u, \Psi u) + \frac{\eta}{2} d(w, \Psi w) < \eta d(u, \Psi u) \leq d(u, \Psi u), \text{for } w \in \Psi u, \end{aligned}$$

$\Rightarrow d(u, \Psi u) < d(u, \Psi u)$, a contradiction, which proves (ii).

(iii) It can be proved from (ii), by using definition of MGN (α, β) mapping.

□

The subsequent results provide a quantitative bound on $H(\Psi u, \Psi v)$ in terms of $d(u, \Psi u)$ and $||u - v||$, thereby reflecting the localized contractive nature of mappings satisfying MGN (α, β) condition and linking their approximate fixed point property with a continuity-like control over the mapping's behavior.

Lemma 3.2. *Let T be a subset (nonempty) of a B.S. $(M, \|.\|)$ and $\Psi : T \to CB(T)$ be an MGN (α, β) mapping, then*

$$H(\Psi u, \Psi v) \leq \frac{(2 - (\alpha^2 + \beta^2))}{(1 - \beta)^2} d(u, \Psi u) + \|u - v\| \text{ for all } u, v \in T.$$

Proof. For $u \in T$, since, Ψu is proximinal, there exists a point $w \in \Psi u$ satisfying $\|w - u\| = d(u, \Psi u)$. Also as $\eta \in [0, 1]$, $\frac{\eta}{2} d(u, \Psi u) < d(u, \Psi u)$. Since, Ψ is an MGN (α, β) mapping, so by definition

$$\begin{aligned} H(\Psi u, \Psi w) &\leq \alpha d(w, \Psi u) + \beta d(u, \Psi w) + (1 - (\alpha + \beta)) \|u - w\|, \\ &= \beta d(u, \Psi w) + (1 - (\alpha + \beta)) \|u - w\| \\ &\leq \beta d(u, \Psi u) + \beta H(\Psi u, \Psi w) + (1 - (\alpha + \beta)) \|u - w\|, \end{aligned}$$

which on using $\|w - u\| = d(u, \Psi u)$ gives,

$$\begin{aligned} H(\Psi u, \Psi w) &\leq \beta \|u - w\| + \beta H(\Psi u, \Psi w) + (1 - (\alpha + \beta)) \|u - w\| \\ &= \beta H(\Psi u, \Psi w) + (1 - \alpha) \|u - w\|, \end{aligned}$$

which gives

$$H(\Psi u, \Psi w) \leq \frac{(1 - \alpha)}{(1 - \beta)} \|u - w\|.$$

From Lemma 3.1, we get $\forall u, v \in T$, either

(i) $H(\Psi u, \Psi v) \leq \alpha d(v, \Psi u) + \beta d(u, \Psi v) + (1 - (\alpha + \beta)) \|u - v\|$, or

(ii) $H(\Psi w, \Psi v) \leq \alpha d(v, \Psi w) + \beta d(w, \Psi v) + (1 - (\alpha + \beta)) \|w - v\|$.

If (i) holds, then

$$\begin{aligned} H(\Psi u, \Psi v) &\leq \alpha d(v, \Psi u) + \beta d(u, \Psi v) + (1 - (\alpha + \beta)) \|u - v\| \\ &\leq \alpha d(v, u) + \alpha d(u, \Psi u) + \beta d(u, \Psi u) + \beta H(\Psi u, \Psi v) \\ &\quad + (1 - (\alpha + \beta)) \|u - v\| \\ &= (\alpha + \beta) d(u, \Psi u) + \beta H(\Psi u, \Psi v) + (1 - \beta) \|u - v\| \end{aligned}$$

$$\Rightarrow H(\Psi u, \Psi v) \leq \frac{(\alpha + \beta)}{(1 - \beta)} d(u, \Psi u) + \|u - v\|.$$

If (ii) holds, we obtain

$$H(\Psi w, \Psi v) \leq \alpha d(v, \Psi w) + \beta d(w, \Psi v) + (1 - (\alpha + \beta)) \|w - v\|.$$

Also,

$$\begin{aligned}
H(\Psi u, \Psi v) &\leq H(\Psi u, \Psi w) + H(\Psi w, \Psi v)\\
&\leq \frac{(1-\alpha)}{(1-\beta)} \|u-w\| + \alpha d(v, \Psi w) + \beta d(w, \Psi v) + (1-(\alpha+\beta)) \|w-v\|\\
&\leq \frac{(1-\alpha)}{(1-\beta)} \|u-w\| + \alpha \|v-u\| + \alpha d(u, \Psi u) + \alpha H(\Psi u, \Psi w) + \beta \|w-u\|\\
&+ \beta d(u, \Psi u) + \beta H(\Psi u, \Psi v) + (1-(\alpha+\beta)) \|w-u\| + (1-(\alpha+\beta)) \|u-v\|\\
&\leq \frac{(1-\alpha)}{(1-\beta)} \|u-w\| + \alpha H(\Psi u, \Psi w) + \beta H(\Psi u, \Psi v) + (1-\alpha) \|w-u\|\\
&+ (1-\beta) \|u-v\| + (\alpha+\beta) \|u-w\| .
\end{aligned}$$

$$\begin{aligned}
\Rightarrow H(\Psi u, \Psi v) &\leq (\frac{(1-\alpha)}{(1-\beta)} + (1-\alpha) + (\alpha+\beta)) \|u-w\| + \alpha H(\Psi u, \Psi w) + \beta H(\Psi u, \Psi v)\\
&+ (1-\beta) \|u-v\|\\
&= \frac{(2-\alpha-\beta^2)}{(1-\beta)} \|u-w\| + \alpha H(\Psi u, \Psi w) + \beta H(\Psi u, \Psi v) + (1-\beta) \|u-v\|\\
&\leq \frac{(2-\alpha-\beta^2)}{(1-\beta)} \|u-w\| + \alpha (\frac{(1-\alpha)}{(1-\beta)} \|u-w\| + \beta H(\Psi u, \Psi v) + (1-\beta) \|u-v\|\\
&= \frac{(2-\alpha^2-\beta^2)}{(1-\beta)} \|u-w\| + \beta H(\Psi u, \Psi v) + (1-\beta) \|u-v\|
\end{aligned}$$

$$\Rightarrow H(\Psi u, \Psi v) \leq \frac{(2-(\alpha^2+\beta^2))}{(1-\beta)^2} d(u, \Psi u) + \|u-v\| .$$

□

The next result shows that for MGN (α, β) maps, proximity to a fixed point ensures continuity-like control over the images of neighboring points, a property essential in the study of iterative convergence.

Lemma 3.3. *Suppose $\Psi : T \to CB(T)$ be an MGN (α, β) map, where $T(\neq \emptyset)$ is a subset of a B.S. $(M, \|.\|)$. Then, $\forall\, u, v, w \in T$*

$$d(u, \Psi v) \leq \Big(\frac{3-2\beta-\alpha^2}{(1-\beta)^2}\Big) d(u, \Psi u) + \|u-v\|$$

Proof. By Lemma 3.1, we have for all $u, v \in T$ and $w \in \Psi u$, either

$$\text{case (i) } H(\Psi u, \Psi v) \leq \alpha d(v, \Psi u) + \beta d(u, \Psi v) + (1-(\alpha+\beta)) \|u-v\| ,$$

$$\text{case (ii) } H(\Psi w, \Psi v) \leq \alpha d(v, \Psi w) + \beta d(w, \Psi v) + (1-(\alpha+\beta)) \|w-v\| .$$

If first case holds true, then we get,

$$\begin{aligned}
d(u, \Psi v) &\leq d(u, \Psi u) + H(\Psi u, \Psi v)\\
&\leq d(u, \Psi u) + \alpha d(v, \Psi u) + \beta d(u, \Psi v) + (1-(\alpha+\beta)) \|u-v\|
\end{aligned}$$

which gives,

$$d(u, \Psi v) \leq \frac{1+\alpha}{1-\beta} d(u, \Psi u) + \|u-v\| .$$

If second holds true, and let $w' \in \Psi u$ be s.t. $\|w' - u\| = d(u, \Psi u)$.

Applying Lemma 3.1 (i) & (iii), it follows that

$$\begin{aligned}
d(u, \Psi v) &\le d(u, \Psi u) + H(\Psi u, \Psi w') + H(\Psi w', \Psi v) \\
&\le d(u, \Psi u) + \frac{1+\alpha}{1-\beta}\|w' - u\| + H(\Psi w', \Psi v) \\
&\le \Big(1 + \frac{1+\alpha}{1-\beta}\Big) d(u, \Psi u) + \alpha d(v, \Psi \mathsf{z}') + \beta d(w', \Psi v) + (1-(\alpha+\beta))\|w' - v\| \\
&\le \Big(1 + \frac{1+\alpha}{1-\beta}\Big) d(u, \Psi u) + \alpha d(v, \Psi u) + \alpha H(\Psi u, \Psi w') + \beta\|w' - u\| + \beta d(u, \Psi v) \\
&+ (1-(\alpha+\beta))\|w' - u\| + (1-(\alpha+\beta))\|u - v\| \\
&\le \Big(1 + \frac{1+\alpha}{1-\beta}\Big) d(u, \Psi u) + \alpha d(v, \Psi u) + \alpha\Big(\frac{1+\alpha}{1-\beta}\Big)\|w' - u\| + \beta\|w' - u\| \\
&+ \beta d(u, \Psi v) + (1-(\alpha+\beta))\|w' - u\| + (1-(\alpha+\beta))\|u - v\|,
\end{aligned}$$

which gives

$$d(u, \Psi v) \le \Big(\frac{3 - 2\beta - \alpha^2}{(1-\beta)^2}\Big) d(u, \Psi u) + \|u - v\|.$$

□

The following theorem extends the classical demiclosed principle to MGN (α, β) maps, establishing that weak limits of approximate fixed point sequences coincide with true fixed points and thus form a fundamental link between weak convergence and fixed point existence.

Theorem 3.1. *Consider $T(\neq \emptyset)$ to be a closed convex subset of a uniformly convex Banach space (UCBS) $(M, \|.\|)$ satisfying (1), $\{u_n\}$ be a sequence in M and $\Psi : T \to CB(T)$ be an MGN (α, β) mapping. If the sequence $\{u_n\}$ is weakly convergent to some point $u \in T$ and $\liminf_{n\to\infty} d(u_n, Su_n) = 0$, then $u \in \Psi u$, or equivalentely $(I - \Psi)$ is demiclosed at zero.*

Proof. For every $n \in \mathbb{N}$, $\exists\, w_n \in \Psi u$ such that $\|u_n - w_n\| = d(u_n, \Psi u)$, since $u \in T$ and Ψu is closed and bounded. Then, by Lemma 3.2,

$$\begin{aligned}
\|u_n - w_n\| = d(u_n, \Psi u) &\le d(u_n, \Psi u_n) + H(\Psi u_n, \Psi u) \\
&\le d(u_n, \Psi u_n) + \frac{(2 - (\alpha^2 + \beta^2))}{(1-\beta)^2} d(u_n, \Psi u_n) + \|u_n - u\|.
\end{aligned}$$

Taking limit infimum on both sides and employing the condition that $\liminf_{n\to\infty} d(u_n, \Psi u_n) = 0$, we get

$$\liminf_{n\to\infty} \|u_n - w_n\| \le \liminf_{n\to\infty} \|u_n - u\|, \forall n \in \mathbb{N}. \tag{3}$$

Using the hypothesis that $\{u_n\}$ converges weakly to u and the Banach space M satisfies (1), for any $n \in \mathbb{N}$, if $w_n \neq u$, we get,

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - w_n\|,$$

which contradicts (3), Thus, we have $w_n = u, \forall n \in \mathbb{N}$.
Also, since, $w_n \in \Psi u$, we have $u \in \Psi u$, which shows that, the operator $(I - \Psi)$ has demiclosedness property at zero. □

3.2 Convergence Results

We give a variant of M-iterative process in this section for finding common fixed points of three MGN (α, β) mappings within the framework of Banach space.

In 2018, Ullah & Arshad [41] defined M-iterative process as: Let $T(\neq \emptyset)$ be a subset of a B.S. $(M, \|.\|)$ and $\Psi : T \to T$ be a mapping, the sequence $\{x_n\}$ defined as below is known as M-iteration scheme.

$$\begin{cases} x_0 \in C, \\ z_n = (1-\alpha_n)x_n + \alpha_n \Psi x_n, \\ y_n = \Psi z_n, \\ x_{n+1} = \Psi y_n. \end{cases} \tag{4}$$

In 2021, Ullah et. al. [42] gave a multivalued version of M-iteration process:

Let $\Psi : T \to CB(T)$ be a multivalued mapping defined on a subset $T(\neq \emptyset)$ of a B.S. $(M, \|.\|)$, the multivalued version of M-iteration process is given by the sequence $\{x_n\}$ defined as:

$$\begin{cases} x_0 \in C, \\ z_n = (1-\alpha_n)x_n + \alpha_n u_n, \\ y_n = w_n, \\ x_{n+1} = v_n, \end{cases} \tag{5}$$

where $u_n \in \Psi x_n$, $w_n \in \Psi z_n$ and $v_n \in \Psi y_n$.

Now, we extend the M-iterative process for three MGN (α, β) maps.

Let $T(\neq \emptyset)$ be a subset of a B.S. $(M, \|.\|)$ and $\Psi_1, \Psi_2, \Psi_3 : T \to CB(T)$ be three MGN (α, β) mappings, the sequence $\{x_n\}$ defined as follows:

$$\begin{cases} x_0 \in C, \\ z_n = (1-\alpha_n)x_n + \alpha_n u_n, \text{where } u_n \in \Psi_1 x_n \\ y_n = w_n \in \Psi_3 z_n, \\ x_{n+1} = v_n \in \Psi_2 y_n, \end{cases} \tag{6}$$

The following lemmas will be used in the subsequent results.

Lemma 3.4. [32] *"Let α_n be a sequence of real numbers in a UCBS $(M, \|.\|)$ such that $0 < p \leq \alpha_n \leq q < 1$ for all $n \in \mathbb{N}$. Consider two sequences $\{u_n\}$ and $\{v_n\}$ in M with $\limsup_{n\to\infty} \|v_n\| \leq k$, $\limsup_{n\to\infty} \|u_n\| \leq k$ and $\limsup_{n\to\infty} \|\alpha_n u_n + (1-\alpha_n)v_n\| \leq k$ for some $k \geq 0$.Then, it follows that $\lim_{n\to\infty} \|u_n - v_n\| = 0$."*

The following lemma establishes the asymptotic regularity of the sequence (6) generated by mappings Ψ_1, Ψ_2, Ψ_3 satisfying condition(3.1) with respect to the common fixed point set.

Lemma 3.5. *Let $\Psi_1, \Psi_2, \Psi_3 : T \to CB(T)$ be defined on a compact convex subset $T(\neq \emptyset)$ of a UCBS $(M, \|.\|)$, satisfying (3.1) such that $F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3) \neq \emptyset$ and $\Psi_i \mathrm{g} = \{\mathrm{g}\}$ $(i = 1, 2, 3)$ $\forall$ $\mathrm{g} \in F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3)$. Let $\{u_n\}$ be the sequence given by (6), then, $\lim_{n\to\infty} \|x_n - \mathrm{g}\|$ exists $\forall \mathrm{g} \in F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3)$.*

Proof. Since, Ψ_1, Ψ_2, Ψ_3 satisfy (3.1) and $g \in F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3)$, by Proposition 3.1, we have, $H(\Psi_1 x_n, \Psi_1 g) \leq \|x_n - g\|$, $H(\Psi_2 y_n, \Psi_2 g) \leq \|y_n - g\|$ and $H(\Psi_3 z_n, \Psi_3 g) \leq \|z_n - g\|$.

Now,

$$\begin{aligned} \|z_n - g\| &= \|(1-\alpha_n)x_n + \alpha_n u_n - g\| \\ &\leq (1-\alpha_n)\|x_n - g\| + \alpha_n \|u_n - g\| \\ &\leq (1-\alpha_n)\|x_n - g\| + \alpha_n H(\Psi_1 x_n, \Psi_1 g) \\ &\leq (1-\alpha_n)\|x_n - g\| + \alpha_n \|x_n - g\| = \|x_n - g\| \end{aligned}$$

$$\Rightarrow \|z_n - g\| \leq \|x_n - g\|.$$

In a similar way,

$$\|y_n - g\| = \|w_n - g\| \leq H(\Psi_3 z_n, \Psi_3 g) \leq \|z_n - g\|$$

$$\Rightarrow \|y_n - g\| \leq \|z_n - g\|.$$

Proceeding as above, we get

$$\|x_{n+1} - g\| = \|v_n - g\| \leq H(\Psi_2 y_n, \Psi_2 g) \leq \|y_n - g\| \leq \|z_n - g\|$$

$$\Rightarrow \|x_{n+1} - g\| \leq \|x_n - g\|$$

Thus, $\|x_n - g\|$ is bounded and non-increasing, and so $\lim_{n\to\infty} \|x_n - g\|$ exists for $g \in F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3)$. □

The lemma given below refines the preceding result by proving the pointwise asymptotic regularity of the sequence (6), ensuring that each iterate approaches an approximate fixed point of the mappings $\Psi_i (i = 1, 2, 3)$.

Lemma 3.6. *Let $M, T, \Psi_i (i = 1, 2, 3)$ be as stated in Lemma 3.5 and $\{x_n\}$ be the sequence given by the M-iteration process (6), then we have*
$\lim_{n\to\infty} d(x_n, \Psi_1 x_n) = \lim_{n\to\infty} d(x_n, \Psi_2 x_n) = \lim_{n\to\infty} d(x_n, \Psi_3 x_n) = 0.$

Proof. Since, Ψ_1, Ψ_2, Ψ_3 are MGN (α, β) mappings and $g \in F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3)$, by Proposition 3.1(iii), we have $H(\Psi_1 x_n, \Psi_1 g) \leq \|x_n - g\|$, $H(\Psi_2 y_n, \Psi_2 g) \leq \|y_n - g\|$ and $H(\Psi_3 z_n, \Psi_3 g) \leq \|z_n - g\|$. From which it follows that $\lim_{n\to\infty} \|x_n - g\|$ exists using Lemma 3.5. Let us assume that

$$\lim_{n\to\infty} \|x_n - g\| = \mathsf{k}. \tag{7}$$

Since,

$$\|u_n - g\| \leq H(\Psi_1 x_n, \Psi_1 g) \leq \|x_n - g\|,$$

$$\|v_n - g\| \leq H(\Psi_2 y_n, \Psi_2 g) \leq \|y_n - g\| \leq \|z_n - g\| \leq \|x_n - g\|,$$

and

$$\|w_n - g\| \leq H(\Psi_3 z_n, \Psi_3 g) \leq \|z_n - g\| \leq \|x_n - g\|,$$

So, $$\limsup_{n\to\infty} \|u_n - g\| \leq \mathsf{k}, \limsup_{n\to\infty} \|v_n - g\| \leq \mathsf{k} \text{ and } \limsup_{n\to\infty} \|w_n - g\| \leq \mathsf{k} \tag{8}$$

Now, since

$$\|x_{n+1} - g\| = \|v_n - g\|,$$

it follows that

$$\mathsf{k} \leq \liminf_{n\to\infty} \|v_n - \mathsf{g}\|, \tag{9}$$

From (8) & (9)

$$\lim_{n\to\infty} \|v_n - \mathsf{g}\| = \mathsf{k}. \tag{10}$$

Now,

$$\begin{aligned}\mathsf{k} = \lim_{n\to\infty} \|v_n - \mathsf{g}\| &\leq H(\Psi_2 y_n, \Psi_2 \mathsf{g}) \leq \|y_n - \mathsf{g}\| \leq \|z_n - \mathsf{g}\| \\ &\leq (1-\alpha_n)\|x_n - \mathsf{g}\| + \alpha_n \|u_n - \mathsf{g}\| \\ &\leq \mathsf{k}.\end{aligned}$$

As a result, we obtain

$$\lim_{n\to\infty} (1-\alpha_n)\|x_n - \mathsf{g}\| + \alpha_n \|u_n - \mathsf{g}\| = \mathsf{k}. \tag{11}$$

Therefore, by (7), (10), (11) and Lemma 3.4, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0$$

$$i.e., \lim_{n\to\infty} d(x_n, \Psi_1 x_n) = 0.$$

Similarly, we get

$$\lim_{n\to\infty} d(x_n, \Psi_2 x_n) = \lim_{n\to\infty} d(x_n, \Psi_3 x_n) = 0.$$

□

The following theorem connects the established asymptotic regularity of the iterative process(6) with its convergence behavior, ensuring that every subsequential limit of the generated sequence is a common fixed point of the mappings.

Theorem 3.2. *Let $M, T, \Psi_i (i = 1, 2, 3)$ be as described in Lemma 3.5 and $\{x_n\}$ be the sequence generated by the iterative process (6). If for any convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to* g*, then* $\mathsf{g} \in F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3)$.

Proof. Let $\lim_{n\to\infty} \|x_{n_i} - \mathsf{g}\| = 0$. Then from Lemma 3.3 and Lemma 3.6, we have

$$\begin{aligned} d(\mathsf{g}, \Psi_1 \mathsf{g}) &\leq \|\mathsf{g} - x_{n_i}\| + d(x_{n_i}, \Psi_1 \mathsf{g}) \\ &\leq \|\mathsf{g} - x_{n_i}\| + \Big(\frac{3-2\beta-\alpha^2}{(1-\beta)^2}\Big) d(x_{n_i}, \Psi_1 x_{n_i}) + \|\mathsf{g} - x_{n_i}\| \\ &= 2\|\mathsf{g} - x_{n_i}\| + \Big(\frac{3-2\beta-\alpha^2}{(1-\beta)^2}\Big) d(x_{n_i}, \Psi_1 x_{n_i}). \end{aligned}$$

Taking limit $n \to \infty$ on both sides it gives, $\lim_{n\to\infty} d(\mathsf{g}, \Psi_1 \mathsf{g}) = 0$, i.e., $\mathsf{g} \in F(\Psi_1)$. Similarly, it can be shown that $\mathsf{g} \in F(\Psi_2)$ and $\mathsf{g} \in F(\Psi_3)$. Hence, we get $\mathsf{g} \in F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3)$.

□

We now prove the strong convergence of the sequence given by the iteration scheme (6) to a common fixed point of three MGN (α, β) maps.

Theorem 3.3. *Let $M, T, \Psi_i (i = 1, 2, 3)$ be as defined in Lemma 3.5. Then the sequence $\{x_n\}$ generated by the iterative process (6) converges strongly to a common fixed point of Ψ_1, Ψ_2 and Ψ_3.*

Proof. Given the compactness of T, a subsequence $\{x_{n_i}\} \subset \{x_n\}$ exists which converges strongly to a point $\mathrm{g} \in T$, i.e., $\lim_{n\to\infty} \|x_{n_i} - \mathrm{g}\| = 0$. By Theorem 3.2, we get $\mathrm{g} \in F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3)$, and hence by Lemma 3.5, $\lim_{n\to\infty} \|x_n - \mathrm{g}\|$ exists and it must be same as $\lim_{n\to\infty} \|x_{n_i} - \mathrm{g}\|$.

$$\therefore \lim_{n\to\infty} \|x_n - \mathrm{g}\| = 0.$$

Thus, the sequence $\{x_n\}$ converges strongly to $\mathrm{g} \in F(\Psi_1) \cap F(\Psi_2) \cap F(\Psi_3)$. □

Now, on putting $\Psi_1 = \Psi_2 = \Psi_3 = \Psi$ in Theorem 3.3 and considering Ψ to be a single-valued mapping the following result of [41] is obtained.

Corollary 3.1. [41] *"Let $T(\neq \emptyset)$ be a compact convex subset of a UCBS $(M, \|.\|)$ and $\Psi : T \to T$ be a Suzuki generalized nonexpansive mapping with $F(\Psi) \neq \emptyset$. For arbitrarily chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (4), then $\{x_n\}$ converges strongly to a fixed point of Ψ."*

Also, by taking $\Psi_1 = \Psi_2 = \Psi_3 = \Psi$ in our proposed iterative process (6) and using Theorem 3.3, we get the following result of [42].

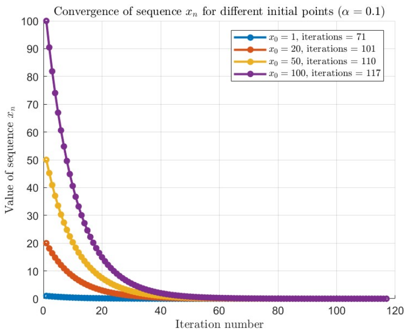
Corollary 3.2. [42] *"Let $\Psi : T \to CB(T)$ be an MGN (α, β) mapping defined on a compact convex subset $T(\neq \emptyset)$ of a UCBS $(M, \|.\|)$. Assume that $F(\Psi) \neq \emptyset$and $\Psi\mathrm{g} = \{\mathrm{g}\}$ for each $\mathrm{g} \in F(\Psi)$. Let $\{x_n\}$ be the sequence generated by (5), then $\{x_n\}$ converges strongly to a fixed point of Ψ."*

Next, we present an example to substantiate the preceding results.

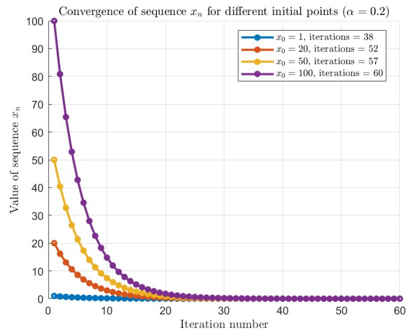
Example 3.3. *Let $M = \mathbb{R}$, $T = [0, \infty)$ and $\Psi_1, \Psi_2, \Psi_3 : T \to CB(T)$ be three multivalued mappings defined by*

$$\Psi_1(x) = \begin{cases} [0, \frac{x}{3}] & \text{if } x \neq 3 \\ [0, 3] & \text{if } x = 3 \end{cases}, \quad \Psi_2(x) = \begin{cases} [0, \frac{x}{4}] & \text{if } x \neq 4 \\ 4 & \text{if } x = 4 \end{cases}, \quad \Psi_3(x) = \begin{cases} [0, \frac{x}{5}] & \text{if } x \neq 5 \\ 3 & \text{if } x = 5 \end{cases}.$$

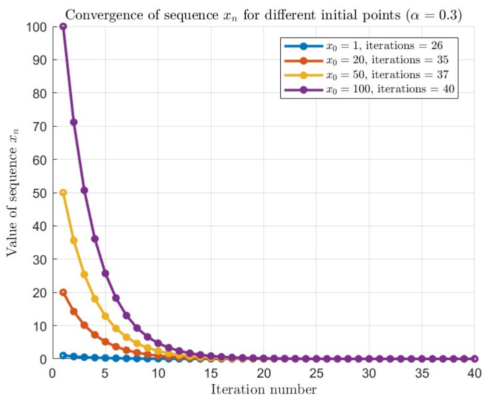
Then, Ψ_1, Ψ_2, Ψ_3 are MGN (α, β) mappings satisfying the conditions of Theorem 3.3, so the sequence $\{x_n\}$ defined by the iterative scheme (6) converges strongly to their common fixed point $\mathrm{g} = 0$.
Fig.1 shows the convergence rate of the iterative sequence $\{x_n\}$ generated by above three mappings Ψ_1, Ψ_2, Ψ_3 for parameter $\alpha_n = 0.1 : 0.1 : 0.9$ on taking different initial values $x_0 = 1, 20, 50, 100$. It can also be observed from Fig. 1 that when the parameters are set very near to zero, the iterative sequence takes more steps to converge.



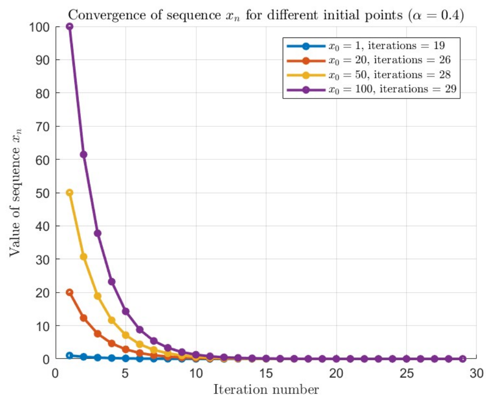
(a) $\alpha_n = 0.1$



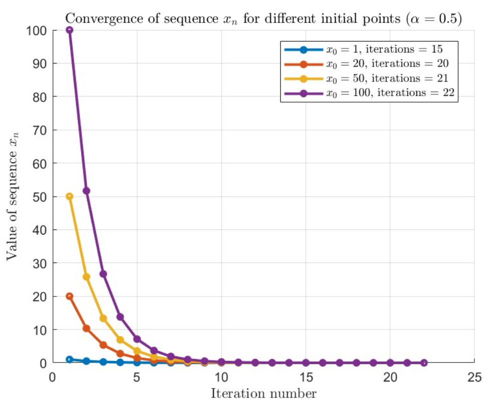
(b) $\alpha_n = 0.2$



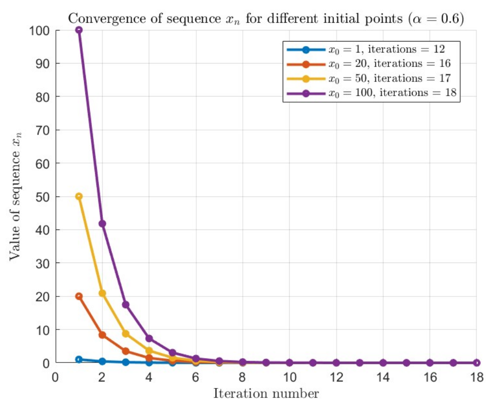
(c) $\alpha_n = 0.3$



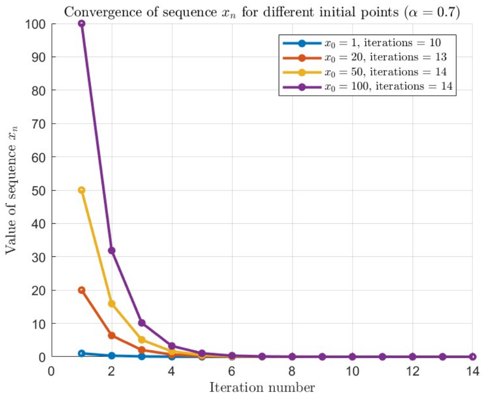
(d) $\alpha_n = 0.4$



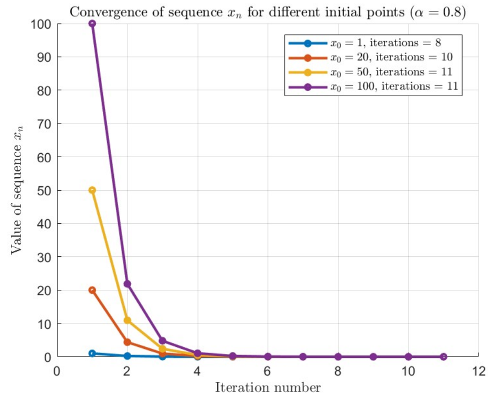
(e) $\alpha_n = 0.5$



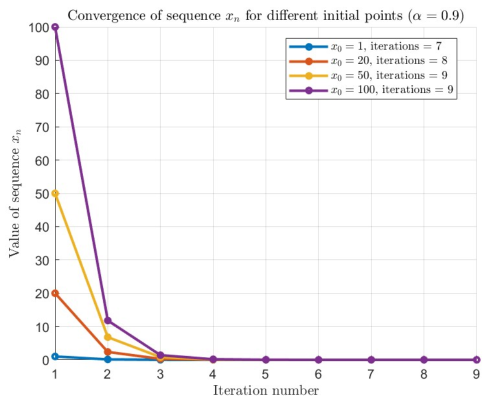
(f) $\alpha_n = 0.6$



(g) $\alpha_n = 0.7$



(h) $\alpha_n = 0.8$



(i) $\alpha_n = 0.9$

Figure 1: Convergence rate of the sequence x_n for parameter $\alpha_n = 0.1 : 0.1 : 0.9$ for different initial values x_0 (Example 3.3).

3.3 Stability Result

Stability of iterative schemes is essential for achieving reliable, accurate, and convergent results which is critical in applications ranging from computational mathematics to engineering and scientific modeling where fixed-point methods are often utilized. Harder [11] defined stability of iterative schemes for single valued maps in metric spaces. The concept of stability of fixed point iterative procedures has been widely studied, beginning with early results by Harder and Hicks [12] and further developed by Berinde [7], among many others. Singh and Chadha [37], utilizing Nadler's theorem [20], generalized Ostrowski's theorem [23] to apply to multivalued contractions, and proposed the following definition of stability for iterative methods involving multivalued mappings in metric spaces.

Let M be a metric space and $\Psi : M \to CL(M)$ be a multivalued mapping. Consider the initial point $u_0 \in M$ and define the Picard iteration by $u_{n+1} \in f(\Psi, u_n)$. Let $\{u_n\}$ be a sequence converging to a fixed point g of the mapping Ψ and consider an arbitrary sequence $\{a_n\} \subset M$. Define $\epsilon_n = H(a_{n+1}, f(\Psi, a_n))$. The iteration $f(\Psi, a_n)$ is said to possess $\Psi-$ stablility whenever

$$\lim_{n\to\infty} \epsilon_n = 0 \implies \lim_{n\to\infty} a_n = \text{g}.$$

The following lemma given by Berinde [7] shall be used in the subsequent result.

Lemma 3.7. *"If δ is a real number such that $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{t_n\}_{n=0}^{\infty}$ satisfying $t_{n+1,} \leq \delta t_n + \epsilon_n,\ \ n = 0, 1, 2, \dots,$ we have, $\lim_{n\to\infty} t_n = 0$."*

Next, we prove that the iterative scheme (6) exhibits Ψ_i- stablity for a contraction map.

Theorem 3.4. *Let $T(\neq \emptyset)$ be a closed convex subset of a UCBS $(M, \|.\|)$. Let $\Psi_i : M \to P(T)(i = 1, 2, 3)$ be three multivalued contraction mappings with $\theta \in (0, 1)$. Let $\{u_n\}$ be a sequence generated by (6) where $\alpha_n \in (0, 1)$, then $\lim_{n\to\infty} ||u_n - \text{g}||$ exists $\forall \text{g} \in \bigcap_{i=1}^{3} F(\Psi_i)$. Then the iterative process (6) is Ψ_i-stable.*

Proof. Nadler's generalization [20] of Banach contraction principle ensures that the mappings $\Psi_i(i = 1, 2, 3)$ have a common fixed point. We will show that u_n converges to some common fixed point of $\Psi_i(i = 1, 2, 3)$. From the iterative process (6)

$$\begin{aligned}
||w_n - \text{g}|| &= (1 - \alpha_n)u_n + \alpha_n u_n - \text{g}|| \\
&\leq ||(1 - \alpha_n)||u_n - \text{g}|| + \alpha_n||u_n - \text{g}|| \\
&\leq ||(1 - \alpha_n)||u_n - \text{g}|| + \alpha_n d(u_n, \Psi_1\text{g}) \\
&\leq ||(1 - \alpha_n)||u_n - \text{g}|| + \alpha_n H(\Psi_1 u_n, \Psi_1\text{g}) \\
&\leq ||(1 - \alpha_n)||u_n - \text{g}|| + \alpha_n\theta||u_n - \text{g}||,
\end{aligned}$$

$$\Rightarrow ||w_n - \text{g}|| \leq (1 - \alpha_n(1 - \theta))||u_n - \text{g}||,$$

$$\begin{aligned}
||v_n - \text{g}|| = ||w_n - \text{g}|| &\leq d(w_n, \Psi_3 g) \leq H(\Psi_3 w_n, \Psi_3\text{g}) \leq \theta||w_n - \text{g}|| \\
&\leq \theta(1 - \alpha_n(1 - \theta))||u_n - \text{g}||.
\end{aligned}$$

Also,

$$\begin{aligned}
||u_{n+1} - \text{g}|| = ||v_n - \text{g}|| \leq d(v_n, \Psi_2 g) \leq H(\Psi_2 v_n, \Psi_2\text{g}) \leq \theta||v_n - \text{g}|| \\
||u_{n+1} - \text{g}|| \leq \theta^2(1 - \alpha_n(1 - \theta))||u_n - \text{g}||.
\end{aligned} \tag{12}$$

By repeating the above process, we get

$$||u_{n+1} - \text{g}|| \leq \theta^2(1 - \alpha_n(1 - \theta))||u_n - \text{g}||,$$

Therefore, we obtain

$$||u_{n+1} - \text{g}|| \leq \theta^{2(k+1)} \prod_{k=0}^{n}(1 - \alpha_k(1 - \theta))||u_0 - \text{g}||.$$

Since, $\theta \in (0, 1)$, $\alpha_n \in (0, 1)$ and $(1 - \alpha_k(1 - \theta) < 1$ for $k = 0, 1, 2, ...,$
Also since, $(1 - u) \leq e^{-u}$, $\forall u \in (0, 1)$, it follows that

$$||u_{n+1} - \text{g}|| \leq \theta^{2(n+1)} e^{-(1-\theta)\sum_{k=0}^{n} \alpha_k}||u_0 - \text{g}||,$$

which shows that $\{u_n\}$ converges to the common fixed point of $\Psi_i(i = 1, 2, 3)$.
Let a_n be any arbitrary sequence in M generated by (6) and the iterative process be given by $a_{n+1} = f(\Psi_i, a_n)$. Set $\epsilon_n = H(a_{n+1}, f(\Psi_i, a_n)(i = 1, 2, 3)$ for $n = 0, 1, 2...,$

To prove that $\lim_{n\to\infty} \epsilon_n = 0 \implies \lim_{n\to\infty} a_n = \mathrm{g}$. Suppose that $lim_{n\to\infty}\epsilon_n = 0$, on using (6) and (12) we get

$$\begin{aligned}
||a_{n+1} - \mathrm{g}|| &\leq ||a_{n+1} - f(\Psi_i, a_n)|| + ||f(\Psi_i, a_n) - \mathrm{g}|| \\
&\leq H(a_{n+1}, f(\Psi_i, a_n)) + \theta^2(1 - \alpha_n(1-\theta))||a_n - \mathrm{g}|| \\
&= \epsilon_n + \theta^2(1 - \alpha_n(1-\theta))||a_n - \mathrm{g}||,
\end{aligned}$$

which on using Lemma 3.7 gives $\lim_{n\to\infty} a_n = \mathrm{g}$. Conversely, let $\lim_{n\to\infty} a_n = \mathrm{g}$ we get,

$$\begin{aligned}
\epsilon_n &= H(a_{n+1}, f(\Psi_i, a_n)) \\
&\leq H(a_{n+1}, \mathrm{g}) + H(\mathrm{g}, f(\Psi_i, a_n)) \\
&\leq ||a_{n+1} - \mathrm{g}|| + ||f(\Psi_i, a_n) - \mathrm{g}|| \\
&\leq ||a_{n+1} - \mathrm{g}|| + \theta^2(1 - \alpha_n(1-\theta))||a_n - \mathrm{g}||
\end{aligned}$$

which tends to 0 as $n \to \infty$ i.e. $\lim_{n\to\infty} \epsilon_n = 0$. Hence, the iterative process (6) is stable with respect to $\psi_i (i = 1, 2, 3)$. □

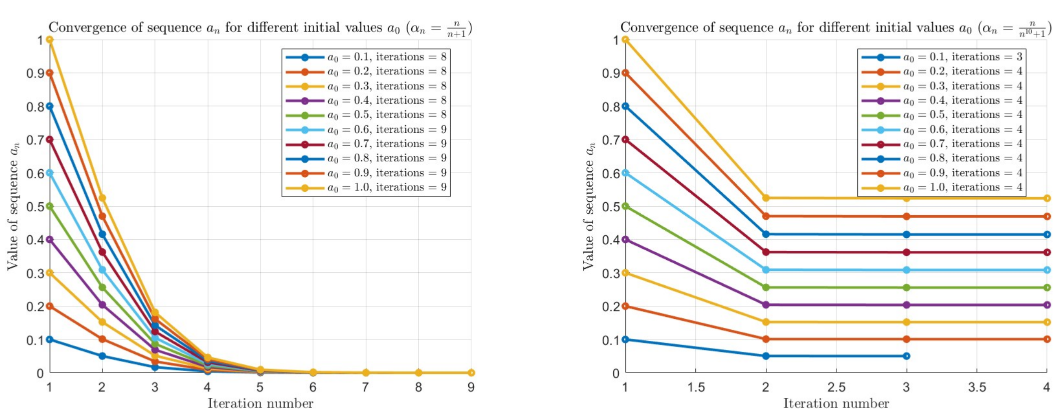
An illustrative example is now provided to support the results discussed above.

Example 3.4. *Let* M=ℝ, $T = [0, \infty)$ *and* $\Psi_1, \Psi_2, \Psi_3 : T \to CB(T)$ *be three multivalued mappings defined by*

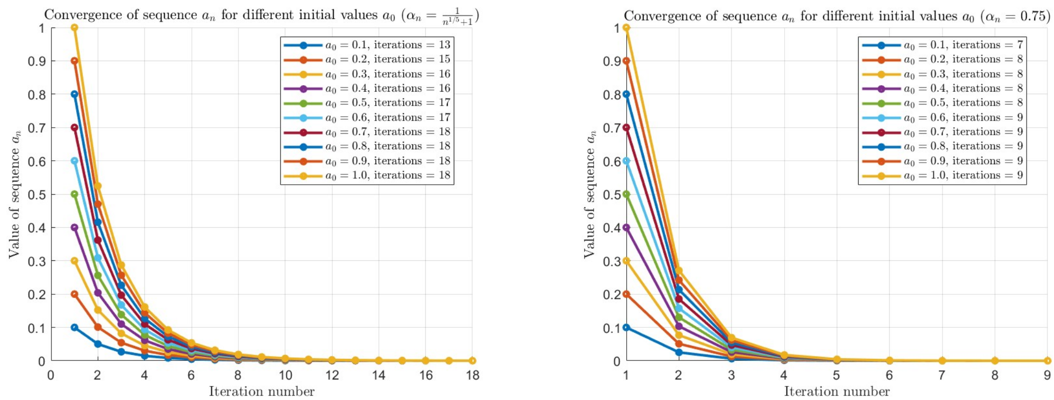
$$\Psi_1(u) = \left\{ \left[0, \tfrac{u}{3}\right] \quad u \in [0,1] \right. , \qquad \Psi_2(u) = \left\{ \left[0, \tfrac{sinu}{3}\right] \quad u \in [0,1] \right. \quad \textit{and}$$

$$\Psi_3(u) = \left\{ \left[0, \frac{u^2}{3}\right] \quad u \in [0,1] \right. .$$

Then, Ψ_1, Ψ_2, Ψ_3 are MGN (α, β) mappings satisfying the conditions of Theorem 3.3, so the sequence $\{u_n\}$ given by the iterative scheme (6) converges strongly to their common fixed point $\mathrm{g} = 0$.
Influence of parameter α_n on the convergence of sequence u_n generated by the above three mappings Ψ_1, Ψ_2, Ψ_3, for different initial values u_0 can be observed from Fig.2. It shows that the iterative method (6) is stable irrespective of the selection of parameter α_n and initial value u_0.



(a) $\alpha_n = \frac{n}{n+1}$ (b) $\alpha_n = \frac{n}{n^{10}+1}$



(c) $\alpha_n = \frac{n}{\sqrt[5]{n}+1}$ (d) $\alpha_n = 0.75$

Figure 2: Influence of parameter α_n on the convergence of sequence u_n for different initial values u_0 (Example 3.4).

4 Application

The concept of Volterra-type integral inclusion extends the classical theory of integral equations to include set-valued functions which finds applications across diverse scientific and engineering disciplines due to their ability to model systems exhibiting memory, feedback, and non-local interactions.
We will use Theorem 3.3 to establish the existence of a solution for the following system of Volterra-type integral inclusion equations

$$u(q) \in\ g(q) + \int_a^q W_i(q, j, u(j))\, dj, \qquad q \in [a, b],\ i = 1, 2, 3 \tag{13}$$

where $W_i : [a, b] \times [a, b] \times \mathbb{R} \rightrightarrows KC(T)$ and $g : [a, b] \to \mathbb{R}$ is a continous function.
Let the space of all continous real valued functions on $[a, b]$ be denoted by $M = C([a, b], \mathbb{R})$. Define the distance function $d(u, t) = \sup_{q\in[a,b]} |u(q) - t(q)|, \forall u \in M$. Then, (M, d) is a metric space.

Theorem 4.1. *Consider the equation* (13) *under the assumptions:*

(i) $W_i : [a, b] \times [a, b] \times \mathbb{R} \to KC(T)$ *and* $g : [a, b] \to \mathbb{R}$ *are a continous function,*W *is compact and convex,*

(ii) for all $s, t \in M$, $m_{i_u}(q, j) \in W_{i_u}(q, j)$, *there exists* $t_{i_v}(q, j) \in W_{i_v}(q, j)$, *such that* $|m_{i_u}(q, j) - t_{i_v}(q, j)| \leq |u - v|$, $i = 1, 2, 3$.

Consequently, a solution $u(q) \in M$ *exists for the system of integral equations* (13).

Proof. Define the multivalued mappings $\Psi_1, \Psi_2, \Psi_3 : C([a, b], \mathbb{R}) \to CP(C([a, b], \mathbb{R}))$ by

$$\Psi_i u(q) = \left\{ s \in M : s \in g(q) + \int_a^q W_i(q, j, u(j))\, dj, \quad q \in [a, b],\ i = 1, 2, 3 \right\}. \tag{14}$$

then u is a solution of system of equations (13) if and only if it is a common fixed point of the multivalued mappings Ψ_1, Ψ_2, Ψ_3.

Assuming that $W_{i_u} = W_i(q, j, u(j))$ for each $q, j \in [a, b]$, then $W_{i_u} : [a, b] \times [a, b] \to KC(T)$, there exists a continous single valued function $m_{i_u} : [a, b] \times [a, b] \to \mathbb{R}$, so that $m_{i_u} \in W_{i_u}$ according to Micheal selection theorem [19]. Consequently, $g(q) + \int_a^q m_{i_u}(q, j)\, dj \in \Psi_i(q)$, which implies $\Psi_i u$ is nonempty and hence is compact.
Now, by assumption (ii), for $i = 1, 2, 3$, there exists $t_{i_v}(q, j) \in W_{i_v}(q, j)$, such that

$$|m_{i_u}(q, j) - t_{i_v}(q, j)| \leq |u - v|\,, q \in [a, b].$$

Let us define the multivalued operator by

$$t(q) = g(q) + \int_a^q t_{i_v}(q, j)\, dj \;\in\; g(q) + \int_a^q W_i(q, j, v(j))\, dj, \qquad q \in [a, b],\; i = 1, 2, 3$$

Then,

$$\begin{aligned} d(s, t) &\leq sup_{q \in [a,b]} \mid u(q) - v(q) \mid \\ &\leq sup_{q \in [a,b]} \left| [g(q) + \int_a^q m_{i_u}(q, j)\, dj] - [g(q) + \int_a^q t_{i_v}(q, j)\, dj] \right| \\ &\leq \mid m_{i_u}(q, j) - t_{i_v}(q, j) \mid \; sup_{q \in [a,b]} \int_a^q dj \\ &\leq |u - v| = d(u, v). \end{aligned}$$

If we interchange the roles of u and v, we obtain $H(\Psi_i u, \Psi_i v) \leq d(u, v)$, which implies that $\Psi_i (i = 1, 2, 3)$ are nonexpansive and hence MGN (α, β) mappings. It follows from Theorem 3.3 that the system of integral equations (13) has a solution obtained through the iterative process (6).

□

5 Conclusion

To approximate common fixed points of three MGN (α, β) mappings in the setting of Banach spaces, we first introduce an extended M-iterative approach. The convergence and stability of the iterative scheme (6) are demonstrated with appropriate assumptions. Our results generalize some of the existing results in the literature. To illustrate the convergence and stability of the proposed approach, several numerical examples are presented.

Acknowledgement We acknowledge and thank the reviewers for their valuable comments which help us to enrich the quality of this paper.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] T. Abdeljawad, K. Ullah, J. Ahmad & N. Mlaiki (2020). Iterative approximation of endpoints for multivalued mappings in banach spaces. *Journal of Function Spaces*, *2020*, 1–5. https://doi.org/10.1155/2020/2179059.

[2] A. Abkar & M. Eslamian (2010). Fixed point theorems for suzuki generalized nonexpansive multivalued mappings in banach spaces. *Fixed Point Theory and Applications*, *2010*, 1–10. https://doi.org/10.1155/2010/457935.

[3] F. Akutsah & O. K. Narain (2021). On generalized (α, β)-nonexpansive mappings in banach spaces with applications. *Nonlinear Functional Analysis and Applications*, *26*(4), 663–684. https://doi.org/10.22771/nfaa.2021.26.04.02.

[4] D. Ali, A. Hussain, E. Karapinar & P. Cholamjiak (2022). Efficient fixed-point iteration for generalized nonexpansive mappings and its stability in banach spaces. *Open Mathematics*, *20*(1), 1753–1769. https://doi.org/10.1515/math-2022-0461.

[5] K. Aoyama & F. Kohsaka (2011). Fixed point theorem for α-nonexpansive mappings in banach spaces. *Nonlinear Analysis: Theory, Methods & Applications*, *74*(13), 4387–4391. https://doi.org/10.1016/j.na.2011.03.057.

[6] S. Banach (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, *3*, 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[7] V. Berinde (2002). On the stability of some fixed point procedures. *Informatica*, *18*(1), 7–14.

[8] L. E. J. Brouwer (1911). Über abbildung von mannigfaltigkeiten. *Mathematische Annalen*, *71*(1), 97–115. https://doi.org/10.1007/BF01456931.

[9] K. Goebel (1975). On a fixed point theorem for multivalued nonexpansive mappings. *Annales Universitatis Mariae Curie-Skłodowska*, *29*, 69–72.

[10] L. Górniewicz (2006). *Topological fixed point theory of multivalued mappings*. Springer, Berlin and Heidelberg.

[11] A. M. Harder (1981). Fixed point theory and stability results for fixed point iteration procedures. *Journal of Mathematical Analysis and Applications*, *81*(2), 205–224. https://doi.org/10.1016/0022-247X(81)90112-1.

[12] A. M. Harder & T. L. Hicks (1988). Stability results for fixed point iteration procedures. *Mathematica Japonica*, *33*(5), 693–706.

[13] A. Hussain, D. Ali & A. H. Albargi (2025). On assessing convergence and stability of a novel iterative method for fixed-point problems. *AIMS Mathematics*, *10*(7), 15333–15357. https://doi.org/10.3934/math.2025687.

[14] M. Iqbal, A. Ali, H. A. Sulami & A. Hussain (2024). Iterative stability analysis for generalized α-nonexpensive mappings with fixed points. *Axioms*, *13*(3), 156. https://doi.org/10.3390/axioms13030156.

[15] T. Ishtiaq, A. Batool, A. Hussain & H. Alsulami (2025). Fixed point approximation of nonexpansive mappings and its application to delay integral equation. *Journal of Inequalities and Applications*, *2025*(1), 19. https://doi.org/10.1186/s13660-025-03263-0.

[16] T. Kaczynski (2008). Multivalued maps as a tool in modeling and rigorous numerics. *Journal of Fixed Point Theory and Applications*, *4*, 151–176. https://doi.org/10.1007/s11784-008-0089-y.

[17] S. Kakutani (1941). A generalization of brouwer's fixed point theorem. *Duke Mathematical Journal*, *8*(3), 457–459. https://doi.org/10.1215/S0012-7094-41-00838-4.

[18] J. T. Markin (1968). A fixed point theorem for set-valued mappings. *Bulletin of the American Mathematical Society*, *74*(4), 639–640. https://doi.org/10.1090/S0002-9904-1968-11971-8.

[19] E. Michael (1966). A selection theorem. *Proceedings of the American Mathematical Society*, *17*(6), 1404–1406. https://doi.org/10.1090/S0002-9939-1966-0203702-5.

[20] S. B. Nadler (1969). A multivalued contraction mapping. *Pacific Journal of Mathematics*, *30*(2), 475–488. https://doi.org/10.2140/pjm.1969.30.475.

[21] G. A. Okeke, A. V. Udo & R. T. Alqahtani (2025). An efficient iterative scheme for approximating the fixed point of a function endowed with condition (bγ, μ) applied for solving infectious disease models. *Mathematics*, *13*(4), 562. https://doi.org/10.3390/math13040562.

[22] Z. Opial (1967). Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bulletin of the American Mathematical Society*, *73*(6), 591–597. https://doi.org/10.1090/S0002-9904-1967-11761-0.

[23] A. M. Ostrowski (1960). *Solutions of equations and systems of equations*. Academic Press, New York.

[24] R. Pant & R. Shukla (2017). Approximating fixed points of generalized α-nonexpansive mappings in banach spaces. *Numerical Functional Analysis and Optimization*, *38*(2), 248–266. https://doi.org/10.1080/01630563.2016.1276075.

[25] B. Panyanak (2007). Mann and ishikawa iterative processes for multivalued mappings in banach spaces. *Computers & Mathematics with Applications*, *54*(6), 872–877. https://doi.org/10.1016/j.camwa.2007.03.012.

[26] P. Patel, B. Saxena, R. Bhardwaj et al. (2024). Fixed point results in hilbert space via altering distance functions. *Malaysian Journal of Mathematical Sciences*, *18*(4). https://doi.org/10.47836/mjms.18.4.10.

[27] R. P. Pathak, S. Dashputre, S. D. Diwan & R. Gupta (2015). On noor-type iteration schemes for multivalued mappings in cat(0) spaces. *Fixed Point Theory and Applications*, *2015*(1), 133. https://doi.org/10.1186/s13663-015-0380-8.

[28] M. Rashid, A. Kalsoom, A. H. Albargi, A. Hussain & H. Sundas (2024). Convergence result for solving the split fixed point problem with multiple output sets in nonlinear spaces. *Mathematics*, *12*(12), 1825. https://doi.org/10.3390/math12121825.

[29] R. Sadhu & C. Nahak (2024). Common fixed points of generalized α-nonexpansive multivalued mappings via modified s-type iteration. *Kragujevac Journal of Mathematics*, *48*(6), 951–966. https://doi.org/10.46793/KgJMat2406.951S.

[30] S. Salisu, P. Kumam & S. Sriwongsa (2022). Strong convergence theorems for fixed point of multi-valued mappings in hadamard spaces. *Journal of Inequalities and Applications*, *2022*(1), 143. https://doi.org/10.1186/s13660-022-02870-5.

[31] K. P. R. Sastry & G. Babu (2005). Convergence of ishikawa iterates for a multivalued mapping with a fixed point. *Czechoslovak Mathematical Journal*, *55*(4), 817–826. https://doi.org/10.1007/s10587-005-0068-z.

[32] J. Schu (1991). Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bulletin of the Australian Mathematical Society*, *43*(1), 153–159. https://doi.org/10.1017/S0004972700028884.

[33] F. Shaddad, M. S. M. Noorani & S. M. Alsulami (2014). Fixed point results in cone metric spaces for multivalued maps. *Malaysian Journal of Mathematical Sciences*, *8*, 83–102.

[34] A. Shaheen, A. Batool, A. Ali, H. A. Sulami & A. Hussain (2024). Recent developments in iterative algorithms for digital metrics. *Symmetry*, *16*(3), 368. https://doi.org/10.3390/sym16030368.

[35] N. Shahzad & H. Zegeye (2009). On mann and ishikawa iteration schemes for multi-valued maps in banach spaces. *Nonlinear Analysis: Theory, Methods & Applications*, *71*(3-4), 838–844. https://doi.org/10.1016/j.na.2008.10.112.

[36] C. Shiau, K.-K. Tan & C. S. Wong (1975). A class of quasi-nonexpansive multivalued maps. *Canadian Mathematical Bulletin*, *18*(5), 709–714. https://doi.org/10.4153/CMB-1975-124-2.

[37] S. L. Singh, C. Bhatnagar & S. N. Mishra (2005). Stability of iterative procedures for multivalued maps in metric spaces. *Demonstratio Mathematica*, *38*(4), 905–916. https://doi.org/10.1515/dema-2005-0415.

[38] Y. Song & H. Wang (2008). Erratum to "mann and ishikawa iterative processes for multivalued mappings in banach spaces"[comput. math. appl. 54 (2007) 872–877]. *Computers & Mathematics with Applications*, *55*(12), 2999–3002. https://doi.org/10.1016/j.camwa.2007.11.042.

[39] T. Suzuki (2008). Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *Journal of Mathematical Analysis and Applications*, *340*(2), 1088–1095. https://doi.org/10.1016/j.jmaa.2007.09.023.

[40] C. Thangaraj, D. Easwaramoorthy, B. Selmi & B. P. Chamola (2024). Generation of fractals via iterated function system of kannan contractions in controlled metric space. *Mathematics and Computers in Simulation*, *222*, 188–198. https://doi.org/10.1016/j.matcom.2023.08.017.

[41] K. Ullah & M. Arshad (2018). Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process. *Filomat*, *32*(1), 187–196. https://doi.org/10.2298/FIL1801187U.

[42] K. Ullah, M. S. U. Khan & M. de la Sen (2021). Fixed point results on multivalued generalized (α, β)-nonexpansive mappings in banach spaces. *Algorithms*, *14*(8), 223. https://doi.org/10.3390/a14080223.